APPENDIX A

MATRIX ALGEBRA

CONTENTS

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A.1 Introduction:

Fundamental to the intelligent use of computer programs for structural analysis and design is at least a basic knowledge of the methodology upon which these programs are based. Since approximately 1958 the language of structural analysis and design has undergone a most profound change. The new language is that of matrix algebra and corresponding to it matrix notation. This is readily obvious to anyone attempting to follow research publications in this area and should serve to motivate the practicing engineer to understand the new language in order to implement the vast amount of new knowledge wisely. Actually the basic methodology for structural analysis has not changed but only become more fundamental and compact. The primary reason for the change in the language is that computers can easily manipulate large blocks of numbers and solve large numbers of simultaneous equations much more directly than they can operate on methods such as moment distribution. Hence, a brief definition of matrix notation, a description of basic operations employing matrix algebra and some example applications of matrix techniques follow. These methods form the basis for STRUDL's internal operation.

Definition and Notation:

A matrix is defined as a rectangular block, or array, of numbers composed of m rows and n columns. For example:

$$[A]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 2 & -8 \\ 1 & -3 \end{bmatrix}$$

Here we have the matrix A composed of $m = 3$ rows and $n = 2$ columns containing six coefficients a_{ij}. The first index on the coefficient within the array defines its row position and the second index defines its column position or $a_{31} = 1$, where a_{31} is the element in the third row and first column.

A.2 Matrix Operations:

I Addition and Subtraction:

Two or more matrices of the same size (those having the same number of rows and columns) may be added or subtracted.

$$[C]_{2\times2} = [A]_{2\times2} + [B]_{2\times2} = \begin{bmatrix} 6 & 1 \\ 2 & -8 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -2 & -6 \end{bmatrix}$$

$$[D]_{2\times2} = [A]_{2\times2} - [B]_{2\times2} = \begin{bmatrix} 6 & 1 \\ 2 & -8 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & -10 \end{bmatrix}$$

or $c_{ij} = a_{ij} + b_{ij}$, $d_{ij} = a_{ij} - b_{ij}$

II Multiplication:

The product of two matrices (A) (B) is equal to a matrix (C) having the same number or rows as (A) and the same number of columns as (B). The product (C) can only exist when the number of rows in (B) is the same as the number of columns in (A). For example:

$$\underset{[A]}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}_{4X2}} \quad \underset{[B]}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}_{2x3}} = \underset{[C]}{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix}_{4x3}}$$

The coefficients c_{ij} in matrix (C) are determined from the relation

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

where n is the number of rows in (B) or the number of columns in (A). As an example, in the above product

$$c_{12} = a_{11}\ b_{12} + a_{12}\ b_{22} \ \underline{\text{or}}\ c_{23} \quad a_{21}\ b_{13} + a_{22}\ b_{23}$$

$$c_{ij} = (i^{th} \text{ row of } [A] \times \text{ the } j^{th} \text{ column of } [B]\ )$$

One way that may facilitate remembering the multiplication procedure is that

Example 1:

Find the product of matrix A and B.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ -2 & 4 \end{bmatrix}_{3\times 2} \qquad B = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}_{2\times 2}$$

$$[A][B] = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -7 & 16 \\ 0 & 11 \\ 8 & 10 \end{bmatrix}_{3\times 2}$$

Example 2:

As a simple example of a matrix formulation illustrative of the multiplication process consider the pin connected truss of Figure A.2a.

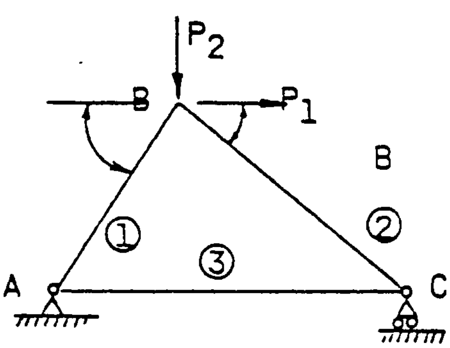


Fig. A.2a

At joint B we have the external forces P_1 and P_2. If we consider the equilibrium of Joint B, Figure A.2b, we have

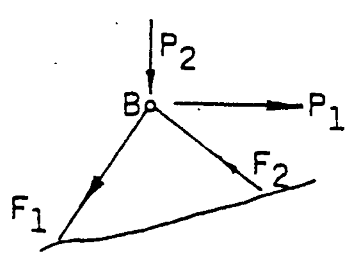


Fig. A.2b

$$\sum F_H = 0$$

or, $P_1 = F_1 \cos \quad + F_2 \cos \beta$ . . . . (1)

$$\sum F_V = 0$$

or, $P_2 = -F_1 \sin\alpha + F_2 \sin \beta$ . . . . (2)

Equations (1) and (2) may be written in matrix form as

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \cos \alpha & \cos \beta \\ -\sin \alpha & \sin \beta \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

If we apply the above role for multiplication we would see that Eq. (3) is the same as Eq.'s (1) and (2), we can write Eq. (3) in an even more abbreviated manner as

$$\{P\} = [A] \quad F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

$$\{P\} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \qquad [A] = \begin{bmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{bmatrix}, \text{ and } \{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Basically then matrix notation, such as Eq. (4), is a short hand way of writing a system of equations.

Example 3:

Suppose we are given the following relationships

$$\begin{aligned} z_1 &= 6y_1 + y_2 \qquad & y_1 &= 3X_1 + 4X_2 \\ z_2 &= 2y_1 - 8y_2 \qquad & y_2 &= 4X_1 + 2X_2 \\ z_3 &= y_1 - 3y_2 \end{aligned}$$

and one wishes to express the z values as a function of the x values. This can be done by direct substitution as follows

$$z_1 = 6\ (3X_1 + 4X_2) + 1\ (-4X_1 + 2X_2) = [\ (6)(3) + (1)(-4)]\ X_1 + [(6)(4)+(1)(2)]\ X_2$$
$$= 14X_1 + 26X_2$$
$$z_2 = 2(3X_1 + 4X_2) - 8(-4X_1 + 2X_2) = [(2)(3) + (-8)(-4)]\ X_1 + [(2)(4)+(-8)(2)]\ X_2$$
$$= -38X_1 - 8X_2$$
$$z_3 = 1(3X_1 + 4X_2) - 3(-4X_1 + 2X_2) = (1)(5)+(-3)(-4)\ X_1 + (1)(4)+(-3)(2)\ X_2$$
$$= 15X_1 - 2X_2$$

This operation also could have been performed using matrix multiplication by writing

$$\{Z\}_{3x1} = [A]_{3x2}\ \{Y\}_{3x1},\quad \{Y\}_{2x1} = [B]_{2x2}\ \{X\}_{2x1}$$

and then by substitution

$$\{Z\}_{3x1} = [A]_{3x2}\ [B]_{2x2} = \{X\}_{2x1} = [C]_{3x2}\ \{X\}_{2x1}$$

Where

$$[C]_{3x2} = [A]_{3x2}\ [B]_{2x2} \text{ and } c_{ij} = \sum_{k=1}^{2} a_{ik}\, b_{kj}$$

Or in general

$$[C]_{LXN}\quad [A]_{LXM}\ [B]_{MXN} \text{ and } c_{ij} = \sum_{k=1}^{M} a_{ik}\, b_{kj}$$

$$[C]_{3x2} = \begin{bmatrix} 6 & 1 \\ 2 & -8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 26 \\ 38 & -8 \\ 15 & -2 \end{bmatrix}$$

$$\therefore \{Z\} = [C]_{3x2}\ \{x\} = \begin{bmatrix} 14x_1 & 26x_2 \\ 38x_1 & -8x_2 \\ 15x_1 & -2x_2 \end{bmatrix}$$

Notice that the results are identical.

III Matrix Inversion:

The inverse, $(A)^{-1}$, of a square matrix (A) is defined such that

$$[A]\,[A]^{-1} = [A]^{-1}\,[A] = [1] \qquad (5)$$

where (I) is defined as the unit matrix. (I) is a square matrix with 1's on the main diagonal and zeros elsewhere. For example, a 3x3 unit matrix is

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can visualize operations with the inverse matrix by a parallel with division i.e., if

$$\begin{array}{ll} & z = xy \\ \text{then} & x = z/y = z(y)^{-1} \\ \text{note that} & (y)(y)^{-1} = Y/Y = 1 \qquad (6) \end{array}$$

observe the similarity of Eq.'s (5) and (6).

Now that we know what the inverse matrix is defined to be, how do we find it. Suppose we are given the set of equations

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3 \end{array} \quad \underline{\text{or}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

and we wish to express the X as function of Y, that is

$$\begin{array}{l} b_{11}y_1 + b_{12}y_2 + b_{13}y_3 = x_1 \\ b_{21}y_1 + b_{22}y_2 + b_{23}y_3 = x_2 \\ b_{31}y_1 + b_{32}y_2 + b_{33}y_3 = x_3 \end{array} \quad \underline{\text{or}} \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

what we are asking is that knowing the coefficients of (A), how may one determine the coefficients in the <u>inverse</u> matrix (B)?

Take as the values of Y, $y_1 = 1$, $y_2 = 0$, $y_3 = 0$, then

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

$$b_{11}(1) + b_{12}(0) + b_{13}(0) = x_1 \text{ , } x_1 = b_{11}$$

$$b_{21}(1) + b_{22}(0) + b_{23}(0) = x_2 \text{ , } x_2 = b_{21}$$

$$b_{31}(1) + b_{32}(0) + b_{33}(0) = x_3 \text{ , } x_3 = b_{31}$$

Therefore, if we determine the values of x by solving the first set of simultaneous equations, we will have the first column of the (B) matrix. If we then take $y_1 = 0$, $y_2 = 1$, $y_3 = 0$ and solve the first set we obtain the solutions constituting the second column of (B). Therefore if we have N equations in N unknowns we must solve N sets of simultaneous equations to obtain the N columns of the (B) matrix.

The above relationship can be written in matrix form as

$$[A]\ \{X\} = \{Y\} \text{ and } [B]\ \{Y\} = \{X\}$$

Notice that if we substitute for (X) in the first of these we obtain

$$[A]\ [B]\ \{Y\} = [A][A]^{-1}\ \{Y\} = \{Y\}$$

and that the only way this relationship can hold is if

$$[A]\ [A]^{-1} = [I]$$

Therefore, if we can find the inverse of a given matrix by some method, we can check its correctness by multiplying the given matrix by its inverse to see if the unit matrix is obtained. The reader may wish to do this to augment his understanding of matrix multiplication and to verify the preceding inversion. Many volumes have been written on different methods of inversion and simultaneous equations solution. These methods will not be discussed here, and it is left to the initiative of the reader to familiarize himself with them.

IV Symmetry and Transposition:

Two additional definitions of particular importance in structural theory employing matrices are the symmetric matrix and the transpose matrix. A symmetric matrix is a square matrix where the coefficients are symmetrical about the main diagonal (i.e., $a_{ij} = a_{ji}$). For example,

$$\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 4 & 15 & 75 \\ 7 & 15 & 3 & 0 \\ 8 & 75 & 0 & 2 \end{bmatrix} \text{— Main Diagonal}$$

A matrix (B) is defined to be the transpose of the matrix (A) if and only if $b_{ij} = a_{ji}$, or

$$[B] = [A]^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -4 & 6 \end{bmatrix} \text{ where } [A] = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 4 & 6 \end{bmatrix}$$

or to put it another way, the rows and columns have been interchanged.

Example 4:

Find the transpose of

$$A = \begin{bmatrix} 1 & 5 & -6 \\ 3 & 8 & 12 \end{bmatrix}$$

Interchanging Rows & Columns:

$$A^T = \begin{bmatrix} 1 & 3 \\ 5 & 8 \\ -6 & 12 \end{bmatrix}$$

A.3 Application to Structural Analysis:

Example 5:

Analyze the truss shown in Figure A.3a by using the principles of matrix algebra.

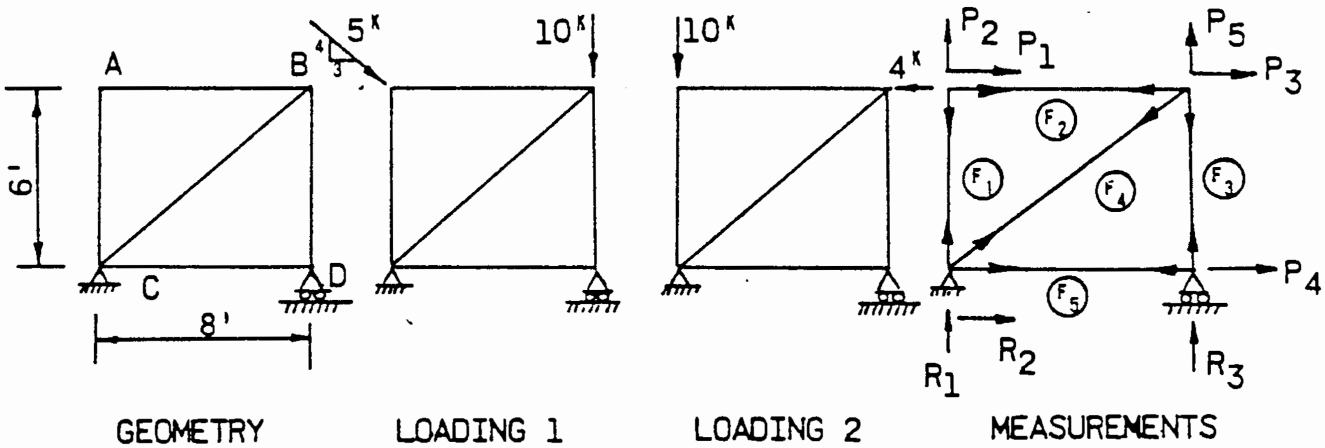


Fig. A.3a

The most fundamental way of analyzing this problem is to write the two equations of equilibrium available at each joint. First observe that any loading system may be defined by two components at joints A and B, which are free to move in any manner, and a component at joint D which is constrained to move only in the horizontal direction. Also note that these five components are sufficient to define any system of applied loads P and if the displacements in the five directions were known, the displaced state of the entire truss would be completely defined. Hence let us refer to this structure as having five degrees of freedom. Additionally there are three constrained displacement quantities and correspondingly three unknown reaction components R_1, R_2, and R_3.

It is important to note that in all structures there will exist this one-to-one correspondence between joint displacements and known joint loads sufficient to totally describe the loading and displacement behavior of the structure. Now, assuming the applied loads and unknown reactions as positive in the sense of the arrows and the unknown bar forces as tension, or pulling on the joints, write the eight available equations of joint equilibrium.

$$
\begin{aligned}
P_1 + F_2 &= 0 \\
P_2 - F_1 &= 0 \\
P_3 - F_2 - 0.8\,F_4 &= 0 \\
P_4 - F_5 &= 0 \\
P_5 - F_3 - 0.6\,F_4 &= 0
\end{aligned}
\qquad
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}
=
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
+1 & 0 & 0 & 0 & 0 \\
0 & +1 & 0 & +0.8 & 0 \\
0 & 0 & 0 & 0 & +1 \\
0 & 0 & +1 & +0.6 & 0
\end{bmatrix}
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}
$$

$$R_1 + F_1 + 0.6\,F_4 = 0$$
$$R_2 + 0.8\,F_4 + F_5 = 0$$
$$R_3 + F_3 = 0$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -0.6 & 0 \\ 0 & 0 & 0 & -0.8 & -1 \\ 0 & 0 & .1 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}$$

In matrix form these can be written as

$$\{P\}_{5\times2} = [A]_{5\times5}\ \{F\}_{5\times2}$$
$$\{R\}_{3\times2} = [A_R]_{3\times5}\ \{F\}_{5\times2}$$

The solution may then be obtained by solving the five simultaneous equations or by the inverse method.

$$\{F\}_{5\times2} = [A]^{-1}_{5\times5}\ \{P\}_{5\times2}$$

The solution of this problem may be found to be

$$\{F\} = [A]^{-1}\ \{P\} = \begin{bmatrix} 0. & +1. & 0. & 0. & 0 \\ -1. & 0. & 0. & 0. & 0 \\ -0.75 & 0. & -0.75 & 0. & +1 \\ 0. & 0. & 0. & +1. & 0 \end{bmatrix} \begin{bmatrix} +3. & 0 \\ -4. & -10 \\ 0. & -4 \\ -10. & 0 \end{bmatrix} = \begin{bmatrix} -4. & -10 \\ -3. & 0 \\ -12.25 & +3 \\ 0. & 0 \end{bmatrix}$$

$$\{R\} = [A_R]\ \{F\} = \begin{bmatrix} -1. & 0. & 0. & -0.6 & 0 \\ 0. & 0. & 0. & -0.8 & -1 \\ 0. & 0. & -1. & 0. & 0 \end{bmatrix} \begin{bmatrix} -4 & -10 \\ -3 & 0 \\ -12.25 & +3 \\ +3.75 & -5 \\ 0. & 0 \end{bmatrix} = \begin{bmatrix} +1.75 & +13 \\ -3. & +4 \\ +12.25 & -3 \end{bmatrix}$$

It is significant to note that the solution to this problem requires only a knowledge of the geometry (defined by the coordinates of the joints) and the magnitude of the applied loads. This is only possible because the system is statically determinate and the solution does not require consideration of the conditions of compatibility. A condition of determinacy is defined when the degree of freedom NP is equal to the number of unknown internal forces NF. This becomes important in using STRUDL when it is desired to carry out an initial approximate determinate analysis in the preliminary design stage of an indeterminate structure. Notice that this problem is the same as Problem 2.2 in Chapter 2.

Example 6:

Next consider the beam shown in the following sketch.

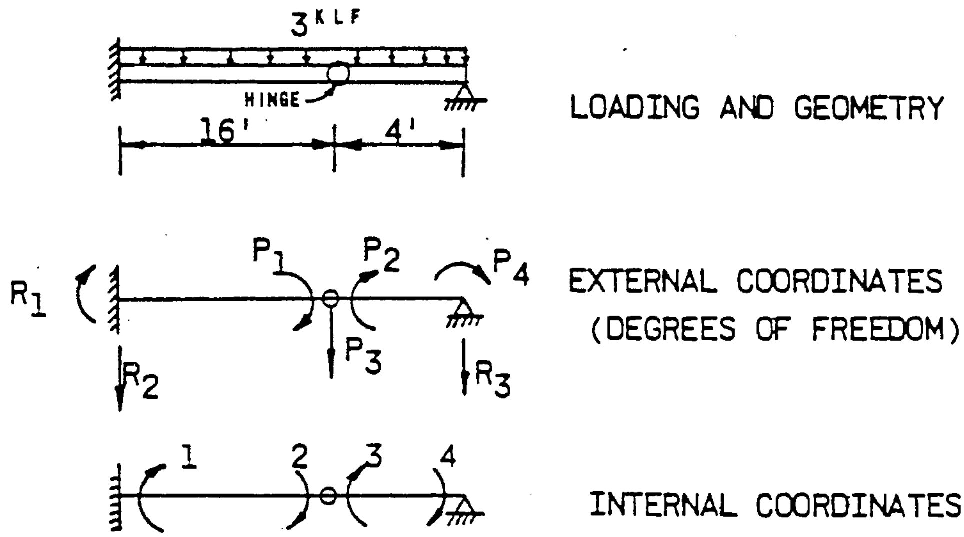
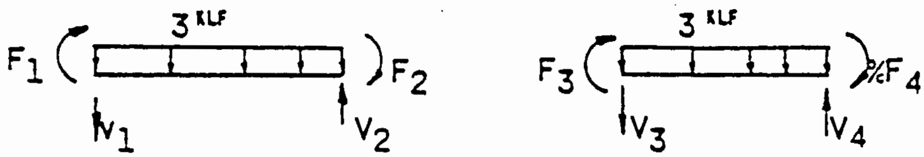


Fig. A.3b

Without the hinge the structure is indeterminate to the first degree and hence the solution would require a knowledge of the beam properties and application of the condition of compatibility and stress-strain. By inserting the hinge the beam is made determinate and the analysis requires only the equations of equilibrium. In general, defining the number of equilibrium equations requires a determination of the degree of freedom. Again this determination may be made by finding the number of displacement quantities required to completely define the end displacements of each element in the structure. In a continuous beam it is necessary to define the end displacements normal to the element axis and the end rotations. In this example there are four unknown displacement quantities required to satisfy this condition, while there are three known displacement quantities associated with the three unknown reaction components.

To write the equations of joint equilibrium it is first necessary to transfer the applied member loads to the ends by looking at element equilibrium.



$$V_1 = \frac{F_1 + F_2}{12} - 18 \qquad V_2 = \frac{F_1 + F_2}{12} + 18$$

$$V_3 = \frac{F_3 + F_4}{4} - 6 \qquad V_4 = \frac{F_3 + F_4}{4} + 6$$

Fig- A.3c

Then joint equilibrium becomes:

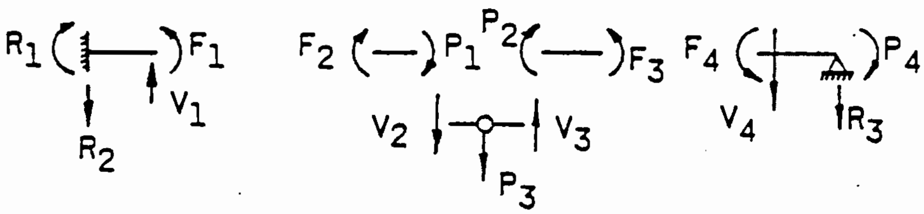


Fig. A.3d

$$\begin{aligned} P_1 + F_2 &= 0 \\ P_2 - F_3 &= 0 \\ P_3 + V_2 - V_3 &= 0 \\ P_4 - F_4 &= 0 \end{aligned} \qquad \begin{bmatrix} P_1 \\ P_2 \\ P_3+24 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0. & 1 & 0. & 0. \\ 0. & 0. & 1. & 0 \\ -1/12 & -1/12 & +1/4 & +1/4 \\ 0. & 0 & -1/4 & -1/4 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

$$\begin{aligned} R_1 - F_1 &= 0 \\ R_2 - V_1 &= 0 \\ R_3 + V_4 &= 0 \end{aligned} \qquad \begin{bmatrix} R_1 \\ R_2+18 \\ R_3+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ +1/12 & +1/12 & 0 & 0 \\ 0 & 0 & -1/4 & -1/4 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

The solution of this problem becomes:

$$[A]^{-1}\{P\} = \{F\} = \begin{bmatrix} +1 & +3 & -12 & +3 \\ -1 & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 1. \end{bmatrix} \begin{bmatrix} 0. \\ 0. \\ 24. \\ 0 \end{bmatrix} = \begin{bmatrix} -288. \\ 0. \\ 0. \\ 0. \end{bmatrix}$$

$$\therefore R^{J}\{F\} = \{R\} = \begin{bmatrix} 1. & 0. & 0. & 0. \\ 1/12 & 1/12 & 0. & 0. \\ 0. & 0. & -1/4 & -1/4 \end{bmatrix} \begin{bmatrix} -288. \\ 0 \\ 0 \\ 0. \end{bmatrix} = \begin{bmatrix} (-288. \; -0\,) \\ (-24. \; -18\,) \\ (\;0. \; -6\,) \end{bmatrix} \begin{bmatrix} -288. \\ -42 \\ -6 \end{bmatrix}$$

Example 7:

Analyze the frame shown in Figure A.3e. Neglect axial deformations.

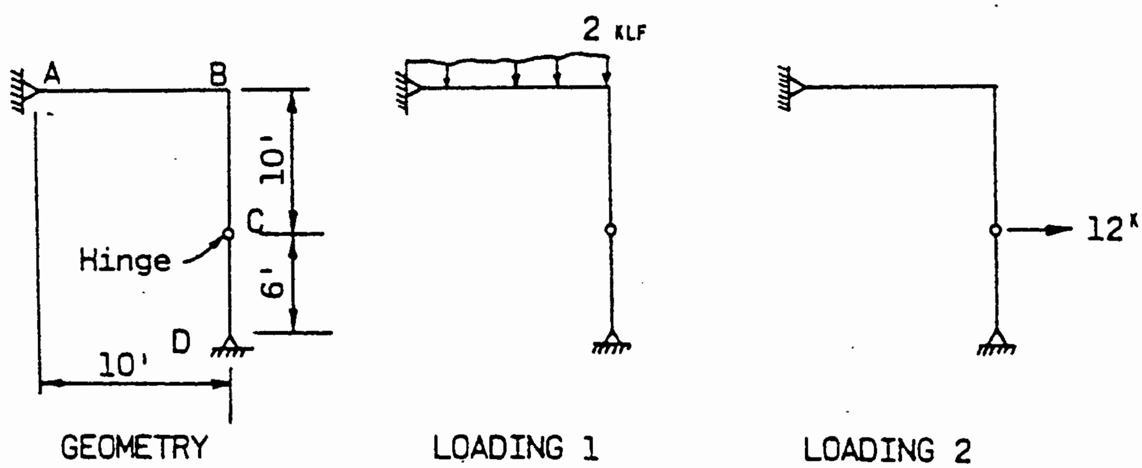
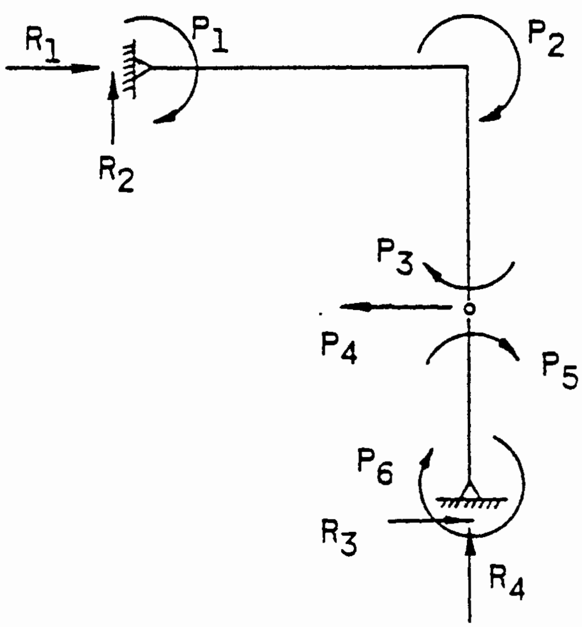


Fig. A.3e



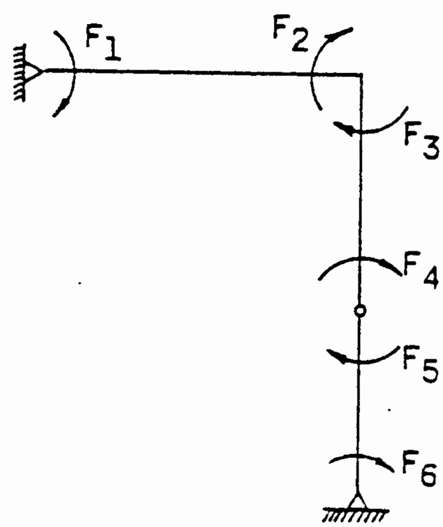
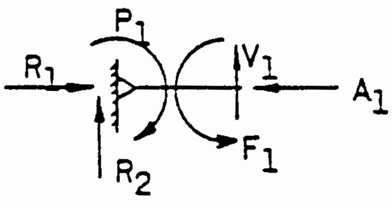


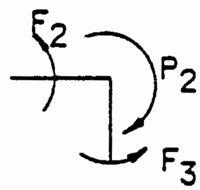
Fig. A.3f

EXTERNAL CO-ORDINATES

Fig. A.3g

INTERNAL CO-ORDINATES

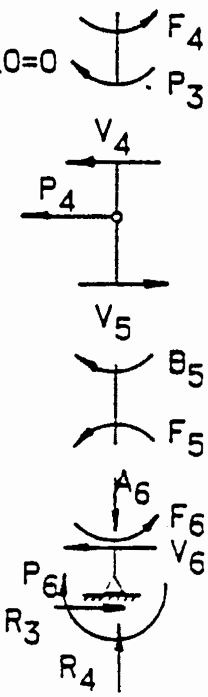




$P_1 - F_1 = 0$

$R_2 + V_1 = 0 \quad -R_2 + \dfrac{-F_1 - F_2}{10} + 10 = 0$

$R_1 - A_1 = 0$



$V_4\ (8) - F_3 - F_4 = 0$

$V_4 = \dfrac{F_8 + F_4}{8}$

$V_5\ (6) - F_5 - F_6 = 0$

$V_5 = \dfrac{F_5 + F_6}{6}$

$V_6 =$

$P_2 - F_2 - F_3 = 0$

$P_3 - F_4 = 0$

$P_4 + V_4 - V_5 = 0$

$P_4 + \dfrac{F_3 + F_4}{8} - \dfrac{F_5 + F_6}{6} = 0$

$P_5 - F_5 = 0$

$P_6 - F_6 = 0$

$R_3 - V_6 = 0 \qquad R_3 - \dfrac{F_5 + F_6}{6} = 0$

$R_4 - A_6 = 0$

Fig. A.3h

$$\left.\begin{aligned} R_1 + R_3 + 0 &= 0 \\ R_2 + R_4 - 20 &= 0 \end{aligned}\right\} \text{LOADING 1}$$

$$\left.\begin{aligned} R_1 + R_3 + 12 &= 0 \\ R_2 + R_4 + 0 &= 0 \end{aligned}\right\} \text{LOADING 2}$$

$P_1 = F_1$

$P_2 = F_2 + F_3$

$P_3 = F_4$

$P_4 = -1/8\ F_3\ -1/8\ F_4\ +1/6\ F_5\ +1/6\ F_6$

$P_5 = F_5$

$P_6 = F_6$

$$\left.\begin{aligned}
&R_1 = 1/6\ F_5\ -1/6\ F_6\\
&R_2\ -10 = 1/10\ F_1\ -1/10\ F_2\\
&R_3 \quad\ =+1/6\ F_5\ +1/6\ F_6\\
&R_4\ -20 = -R_2 = 1/10\ F_1 + 1/10\ F_2\ -10 \longrightarrow R_4\ -10 = 1/10\ F_1\ +1/10\ F_2
\end{aligned}\right\}\ \text{CASE 1}$$

$$\left.\begin{aligned}
&R_1\ +12 = -1/6\ F_5\ -1/6\ F_6\\
&R_2 = -1/10\ F_1\ -1/10\ F_2\\
&R_3 = 1/6\ F_5\ +1/6\ F_6\\
&R_4 = 1/10\ F_1\ +1/10\ F_2
\end{aligned}\right\}\ \text{LOADING 2}$$

$$\{P\} = [A]\{F\}$$

$$\begin{bmatrix} P_1\\ P_2\\ P_3\\ P_4\\ P_5\\ P_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1/8 & -1/8 & +1/6 & +1/6\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1\\ F_2\\ F_3\\ F_4\\ F_5\\ F_6 \end{bmatrix}$$

REACTIONS

$$R = [A_R]\, F$$

LOADING 1

$$\begin{bmatrix} R_1 \\ R_2+10 \\ R_3 \\ R_4-30 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -1/6 & -1/6 \\ -1/10 & -1/10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 \\ 1/10 & 1/10 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 8 & -8/6 & -8/6 \\ 0 & 0 & -1 & -8 & +8/6 & +8/6 \\ 0 & 0 & 11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\{F\} = [A]^{-1}\{P\}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 8 & -1.33 & -1.33 \\ 0 & 0 & -1 & -8 & +1.33 & +1.33 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\{F\} = \{0\}$$

$R_1 = 0$

$R_2 - 10 = 0$ $\qquad$ $R_2 = 10$

$R_3 = 0$ $\qquad$ $R_4 = 10$

$R_4 - 10$

LOADING 2

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 8 & -1.33 & -1.33 \\ 0 & 0 & -1 & -8 & +1.33 & +1.33 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -96 \\ +96 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 + 12 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1/6 & -1/6 \\ -1/10 & -1/10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 \\ 1/10 & 1/10 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -96 \\ +96 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 9.6 \\ 0 \\ -9.6 \end{bmatrix}$$

$R_1 = -12$ $\qquad$ $R_3 = 0$

$R_2 = 9.6$ $\qquad$ $R_4 = -9.6$